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Kinematical Similarity and Exponential Dichotomy of Linear Abstract Impulsive Differential Equations

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The notions of kinematical similarity and exponential dichotomy for linear abstract differential equations are extended to impulsive equations. The fundamental properties of these notions for Banach and Hilbert spaces are investigated.

1. INTRODUCTION

The work of V. D. Mil'man and A. D. Myshkis (1960) initiated the theory of impulsive differential equations. In recent years this theory has undergone rapid development. The monographs of Samoilenko and Perestyuk (1987), Bainov and Simeonov (1989), and Lakshmikantham *et al.* (1989) were starting points for new investigations devoted to this subject.

Investigations devoted to abstract impulsive differential equations have recently appeared (Zabreiko *et al.*, 1988; Bainov *et al.*, 1988*a–c*, 1989*a–c*, 1990, n.d.-*a,b*). In the present paper these investigations are continued and numerous properties of the notions *kinematical similarity* and *exponential dichotomy* are considered.

2. STATEMENT OF THE PROBLEM

Let X be a Banach space with identity operator I. By $L(X)$ we shall denote the space of all linear bounded operators mapping X into X. Let J be a subinterval of $\mathbb{R} = (-\infty, \infty)$ which contains $[0, \infty)$.

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Consider the impulsive differential equation

$$\frac{dx}{dt} = A(t)x \qquad (t \,\bar{\in}\, \{t_n\}) \tag{1}$$

$$x(t_n^+) = Q_n x(t_n) \tag{2}$$

We shall say that Condition A is met if the following conditions hold:
A1. The sequence $T = \{t_n\}_{n=-\infty}^{\infty}$ satisfies

$$t_n < t_{n+1} \quad (n \in \mathbb{Z}), \qquad \lim_{n \to \pm\infty} t_n = \pm\infty$$

A2. $A(t) \in L(X)$, $t \in J \backslash \{t_n\}$, and $A(\cdot)$ is a continuous function on each nonempty interval $[t_n, t_{n+1}] \cap J (n \in \mathbb{Z})$.

A3. $Q_n \in L(X)$ for all n for which $t_n \in J$.

Definition 1. A *solution* of equation (1), (2) we shall call a function $x(t)$ with values in X which is differentiable and satisfies for $t \neq t_n$ equation (1), for $t = t_n$ has discontinuities of the first kind, and is continuous from the left and meets the condition of a "jump" (2).

An operator-valued function $U(t, \tau)$ associating with each element $x_\tau \in X$ $(t, \tau \in J, \tau \leq t)$ just one solution $x(t) = U(t, \tau)x_\tau$ of (1), (2) for which $x(\tau) = x_\tau$ is said to be an *evolutionary operator* of (1), (2) (Zabreiko *et al.*, 1988; Bainov *et al.*, 1989*b*).

It is not hard to check that for $\tau, t \in J$ the following equalities are valid:

$$U(t^-, \tau) = U(t, \tau^-) = U(t, \tau) \qquad (\tau < t)$$

$$U(t, t) = I$$

$$U(t, s) = U(t, \tau)U(\tau, s) \qquad (\tau \leq s \leq t)$$

$$U'_t(t, \tau) = A(t)U(t, \tau) \qquad (\tau \leq t, t \,\bar{\in}\, \{t_n\})$$

$$U'_\tau(t, \tau) = -U(t, \tau)A(\tau) \qquad (t \geq \tau \,\bar{\in}\, \{t_n\})$$

$$U(t_n^+, \tau) = Q_n U(t_n, \tau) \qquad (\tau \leq t_n)$$

For $\tau = 0$ instead of $U(t, 0)$ we shall write $U(t)$.

Lemma 1. Let Condition A hold. Then the evolutionary operator $U(t, \tau)$ $(\tau \leq t;\ \tau, t \in J)$ has the form

$$U(t, \tau) = \begin{cases} U_0(t, \tau), & t_n < \tau \leq t \leq t_{n+1} \\ U_0(t, t_n)Q_n U_0(t_n, t_{n-1}) \cdots Q_{k+1} U_0(t_{k+1}, t_k) Q_k U_0(t_k, \tau) \\ \quad (t_{k-1} < \tau \leq t_k \leq t_n < t \leq t_{n+1}) \end{cases}$$

where $U_0(t, \tau)$ $(\tau \le t; \tau, t \in J)$ is the evolutionary operator of the equation

$$\frac{dx}{dt} = A(t)x$$

Lemma 1 is proved by a straightforward verification.

We shall say that Condition B is met if the following holds:

B. The operators Q_n have bounded inverse operators.

Lemma 2. Let Conditions A and B hold. Then the evolutionary operator $U(t, \tau)$ is defined for all $t, \tau \in J$ and for $t < \tau$ has the form

$$U(t,\tau) = \begin{cases} U_0(t,\tau) \qquad (t_n < t \le \tau \le t_{n+1}) \\ U_0(t, t_n)Q_n^{-1}U_0(t_n, t_{n+1})\cdots Q_{k-1}^{-1}U_0(t_{k-1}, t_k)Q_k^{-1}U_0(t_k, \tau) \\ \quad (t_{n-1} < t \le t_n \le t_k < \tau < t_{k+1}) \end{cases}$$

Lemma 2 is proved by a straightforward verification. ■

3. MAIN RESULTS

3.1. Impulsive Equations in a Banach Space

Let X_1 and X_2 be two nonzero subspaces of X. Set

$$S_n(X_1, X_2) = \inf_{x_1 \in X_1\setminus\{0\}, x_2 \in X_2\setminus\{0\}} \left\| \frac{x_1}{\|x_1\|} + \frac{x_2}{\|x_2\|} \right\|$$

Lemma 3 (Daleckii and Krein, 1974, Chapter IV, Lemma 1.1). Let X split into a direct sum of nonzero subspaces X_1 and X_2, i.e., $X = X_1 \dot{+} X_2$, with projectors P_1 and P_2, respectively, and $X_1 = P_1 X$, $X_2 = P_2 X$, $P_1 + P_2 = I$.

Then the following estimate is valid:

$$\frac{1}{\|P_k\|} \le S_n(X_1, X_2) \le \frac{2}{\|P_k\|} \qquad (k = 1, 2)$$

Definition 2. Let conditions A and B hold. The subspace $Y_1 \subset X$ is said to *induce an exponential dichotomy* for the impulsive equation (1), (2) on the interval J if there exists a subspace $Y_2 \subset X$ such that $X = Y_1 \dot{+} Y_2$ and the following conditions are met:

(i) For the solutions $x_1(t) = U(t)x_0^1$ of (1), (2) with $x_0^1 \in Y$ the estimate

$$\|x_1(t)\| \le N_1 e^{-\nu_1(t-s)}\|x_1(s)\| \qquad (s \le t; s, t \in J)$$

is valid, where $\nu_1, N_1 > 0$ are constants independent of x_0^1.

(ii) For the solutions $x_2(t) = U(t)x_0^2$ of (1), (2) with $x_0^2 \in Y_2$ the estimate

$$\|x_2(t)\| \leq N_2 e^{-\nu_2(s-t)}\|x_2(s)\| \qquad (t \leq s; t, s \in J)$$

is valid, where $\nu_2, N_2 > 0$ are constants independent of x_0^2.

(iii) $S_n(X_1(t), X_2(t)) \geq \gamma > 0$ $(t \in J)$, where $X_k(t) = U(t)Y_k$ $(k = 1, 2)$ and γ is a constant.

Remark 1. From Lemma 3 it follows that condition (iii) is equivalent to uniform boundedness of the projector-valued functions $P_k(t) = U(t)P_kU^{-1}(t)$ $(t \in J; k = 1, 2)$, where P_k are the projectors corresponding to the splitting $X = Y_1 \dot{+} Y_2$, i.e., $P_kX = Y_k$ $(k = 1, 2)$, $P_1 + P_2 = I$.

Definition 3. The impulsive equation (1), (2) is said to be *exponentially dichotomous* on J if there exists a subspace $Y_1 \subset X$ which induces an exponential dichotomy for (1), (2).

Lemma 4. Let Conditions A and B hold for equation (1), (2). Then (1), (2) is exponentially dichotomous if and only if there exist projectors P_1 and P_2 $(P_1 + P_2 = I)$ and positive constants $\tilde{N}_1$, $\tilde{N}_2$, ν_1 and ν_2 such that

$$\|U(t)P_1U^{-1}(s)\| \leq \tilde{N}_1 e^{-\nu_1(t-s)} \qquad (s \leq t; s, t \in J)$$

$$\|U(t)P_2U^{-1}(s)\| \leq \tilde{N}_2 e^{-\nu_2(s-t)} \qquad (t \leq s; s, t \in J)$$

The *proof* of Lemma 4 is a simple modification of the proof of Lemma 3.1, Chapter IV, of Daleckii and Krein (1974).

Lemma 5. Let the following conditions hold:

1. Conditions A and B are met for equation (1), (2).
2. The subspace $Y_1 \subset X$ induces an exponential dichotomy for equation (1), (2), i.e, there exists a subspace $Y_2 \subset X$ such that $X = Y_1 \dot{+} Y_2$ and conditions (i)–(iii) are met.
3. Let $\tilde{Y}_2$ be a subspace of Y for which $X = Y_1 \dot{+} Y_2$ and for which

$$\|U(t)\tilde{P}_2U^{-1}(t)\| \leq M \qquad (t \in J)$$

where $\tilde{P}_2$ is the projector corresponding to $\tilde{Y}_2$, i.e.,

$$\tilde{P}_1X = Y_1, \qquad \tilde{P}_2X = \tilde{Y}_2, \qquad \tilde{P}_1 + \tilde{P}_2 = I$$

Then for the space $\tilde{Y}_2$ condition (ii) is met with possibly another constant $\tilde{N}_2$.

The *proof* of Lemma 5 is a modification of the proof of Remark 3.4, Chapter IV, of Daleckii and Krein (1974).

Consider the impulsive equation

$$\frac{dy}{dt} = \tilde{A}(t)y \qquad (t \bar{\in} \{t_n\}) \tag{3}$$

$$y(t_n^+) = \tilde{Q}_n y(t_n) \tag{4}$$

Let Condition A hold for equation (3), (4).

Definition 4. The impulsive equations (1)–(2) and (3)–(4) are said to be *kinematically similar* if there exists an operator-valued function $S(t)\colon J \to L(X)$ for which the following conditions are valid.

1. $S(t)$ is uniformly bounded ($t \in J$).
2. $S(t)$ is continuous for $t \bar{\in} \{t_n\}$.
3. $S(t)$ has points of discontinuity of the first kind at $t = t_n$, where it is continuous from the left.
4. $S(t)$ has a bounded inverse one ($t \in J$).
5. If $x(t)$ is a solution of (1), (2), then $S(t)x(t)$ is a solution of (3), (4) and vice versa.

Lemma 6. Let Conditions A and B hold for the impulsive equations (1)–(2) and (3)–(4). Then for (1)–(2) and (3)–(4) to be kinematically similar it is necessary and sufficient for there to exist a bounded operator-valued function $S\colon J \to L(X)$ which is differentiable for $t \bar{\in} \{t_n\}$, continuous from the left, has a bounded inverse one, and is a solution of the operator impulsive equation

$$S' = \tilde{A}(t)S - SA(t) \qquad (t \bar{\in} \{t_n\},\ t \in J) \tag{5}$$

$$S(t_n^+) = \tilde{Q}_n S(t_n) Q_n^{-1} \tag{6}$$

Moreover, for the evolutionary operators $U(t)$ of (1)–(2) and of (3)–(4) the following formula is valid:

$$S(t) = \tilde{U}(t)S(0)U^{-1}(t) \tag{7}$$

The *proof* of Lemma 6 is trivial.

Theorem 1. Let the following conditions hold:

1. For equation (1), (2) Conditions A and B hold.
2. Equation (1), (2) is exponentially dichotomous.
3. Equations (1)–(2) and (3)–(4) are kinematically similar with transforming function $S(t)$.

Then equation (3), (4) is also exponentially dichotomous.

Proof. Let $U(t)$, respectively $\tilde{U}(t)$, be the evolutionary operators of (1)–(2) and (3)–(4). By (7) the following equality is valid:

$$\tilde{U}(t) = S(t)U(t)S^{-1}(0)$$

Set $\tilde{P}_k = S(0)P_kS^{-1}(0)$ ($k = 1, 2$; $\tilde{P}_1 + \tilde{P}_2 = I$). Then

$$\tilde{U}(t)P_k\tilde{U}^{-1}(s) = S(t)U(t)P_kU^{-1}(s)S(s) \qquad (k = 1, 2)$$

The assertion of Theorem 1 follows from the above equality and Lemma 4. Theorem 1 is proved. ■

3.2. Impulsive Equations in a Hilbert Space

In this section we assume that X is a real Hilbert space with scalar product $(\cdot\,, \cdot)$.

Lemma 7. Let the subspace Y_1 of the space X induce an exponential dichotomy.

Then without loss of generality we can assume that the projector $P_1: X \to Y_1$ (which enters the definition of dichotomy) is Hermitian (i.e., self-adjoint).

Proof. Let Y_2 be the orthocomplement of Y_1. Then the projector $P_1: X \to Y_1$ with kernel $\ker(P_1) = Y_2$ is the Hermitian operator sought. ■

Theorem 2. Let the following conditions hold:

1. The Hermitian projectors P_1 and P_2 are given and $P_1 + P_2 = I$ and $X = X_1 \dot{+} X_2$, where $X_k = P_kX$ ($k = 1, 2$).
2. For equation (1), (2) Conditions A and B hold.
3. There exists a constant $M > 0$ for which

$$\|U(t)P_kU^{-1}(t)\| \le M \qquad (k = 1, 2; t \in J)$$

Then the impulsive equation (1), (2) is kinematically similar to the impulsive equation with function $\tilde{A}(t)$ and impulsive operators $\tilde{Q}_n$ which commute with the projectors P_k ($k = 1, 2$) and, moreover, the following estimate is valid:

$$\|\tilde{A}(t) + \tilde{A}^*(t)\| \le \|A(t) + A^*(t)\| \qquad (t \in J) \tag{8}$$

Proof. Set

$$R^2(t) = P_1U^*(t)U(t)P_1 + P_2U^*(t)P_2 \tag{9}$$

It is not hard to check that the operator $R^2(t)$ for each $t \in J$ is uniformly positive and Hermitian. Let $R(t)$ be a positive square root of $R^2(t)$. We recall that from the general Poincaré–Riesz formula there follows the representation

$$R(t) = -\frac{1}{2\pi i}\oint_{\Gamma_t} \sqrt{\lambda}[R^2(t) - \lambda I]^{-1}\, d\lambda \tag{10}$$

where Γ_t is a smooth contour which continuously depends on t, encircles the spectrum of the operator $R^2(t)$, lies in the half-plane $\operatorname{Re}\lambda > 0$, and passes in the positive direction. By $\sqrt{\lambda}$ we denote the principal value of the root. The operator $R(t)$ is also Hermitian. Since $R^2(t)$ commutes with P_k, then by (10), $R(t)$ also commutes with P_k ($k = 1, 2$).

Consider the operator-valued function $S(t) = R(t)U^{-1}(t)$ ($t \in J$). Taking into account formula (10), it is not hard to check that the operator-valued functions $S(t)$ and $S^{-1}(t)$ are differentiable for $t \bar{\in} \{t_n\}$, have points of discontinuity of the first kind, at $t = t_n$ and are continuous from the left ($t \in J$).

We shall show that the function $S(t)$ ($t \in J$) is bounded. From the equalities

$$S^*S = (U^{-1})^*R^2U^{-1} = (UP_1U^{-1})^*(UP_1U^{-1}) + (UP_2U^{-1})^*UP_2U^{-1}$$

there follows the estimate

$$\|S\|^2 \le \|UP_1U^{-1}\|^2 + \|UP_2U^{-1}\| \le M^2 + M^2 < \infty$$

We shall show that the function $S^{-1} = UR^{-1}$ is also bounded indeed,

$$\begin{aligned} &P_1(S^{-1})^*S^{-1}P_1 + P_2(S^{-1})^*S^{-1}P_2 \\ &= P_1R^{-1}U^*UR^{-1}P_1 + P_2R^{-1}U^*UR^{-1}P_2 \\ &= R^{-1}(P_1U^*UP_1 + P_2U^*UP_2)R^{-1} = I \end{aligned}$$

Let $z \in X$ be an arbitrarily chosen element. Then

$$\begin{aligned} \|S^{-1}z\|^2 &= \|S^{-1}P_1z + S^{-1}P_2z\|^2 \\ &\le 2\|S^{-1}P_1z\|^2 + 2\|S^{-1}P_2z\|^2 \\ &= 2(S^{-1}P_1z, S^{-1}P_1z) + 2(S^{-1}P_2z, S^{-1}P_2z) \\ &= 2(P_1(S^{-1})^*S^{-1}P_1z, z) + 2(P_2(S^{-1})^*S^{-1}P_2z, z) \\ &= 2(z, z) = 2\|z\|^2 \end{aligned}$$

Choose the function S as a function of kinematical similarity. Then by formula (5)

$$\tilde{A} = SAS^{-1} + S'S^{-1}$$

i.e., taking into account the formula $(U^{-1})' = -U^{-1}U'U^{-1} = -U^{-1}A$, we get

$$\tilde{A} = RU^{-1}AUR^{-1} + (R'U^{-1} - RU^{-1}A)UR^{-1} = R'R^{-1} \tag{11}$$

For the impulsive operators, by formula (6) we obtain

$$\begin{aligned}\tilde{Q}_n = S(t_n^+)Q_nS^{-1}(t_n) &= R(t_n^+)U^{-1}(t_n)Q_n^{-1}Q_nU(t_n)R^{-1}(t_n)\\ &= R(t_n^+)R^{-1}(t_n)\end{aligned}$$

The evolutionary operator of the new impulsive equation is $\tilde{U}(t) = S(t)U(t) = R(t)$. It is not hard to check that the operators $\tilde{A}(t)$ and $\tilde{Q}_n$ commute with P_k $(k = 1, 2)$.

We shall prove inequality (8). Differentiate equality (9):

$$RR' + R'R = P_1U^*(A^* + A)UP_1 + P_2U^*(A^* + A)UP_2$$

For $z \in X$ the following equality is valid:

$$((RR' + R'R)z, z) = ((A^* + A)UP_1z, UP_1z) + ((A^* + A)UP_2z, UP_2z) \tag{12}$$

For each $t \in J$ set $\beta(t) = \|A^*(t) + A(t)\|$. From (12) we obtain the estimate

$$\begin{aligned}((RR' + R'R)z, z) &\le \beta(t)\{(UP_1z, UP_1z) + (UP_2z, UP_2z)\}\\ &= \beta(t)(R^2z, z) = \beta(t)\|Rz\|^2 \qquad (z \in X, t \in J)\end{aligned}$$

From this, taking into account the equality $RR' + R'R = R(\tilde{A} + \tilde{A}^*)R$ [see (11)] and setting $v = Rz$, we obtain the inequality

$$((\tilde{A}(t) + \tilde{A}^*(t))v, v) \le \beta(t)\|v\|^2$$

i.e.,

$$\|\tilde{A}(t) + \tilde{A}^*(t)\| \le \beta(t) = \|A(t) + A^*(t)\| \qquad (t \in J)$$

Theorem 2 is proved. ∎

Remark 2. If the conditions of Theorem 2 are met, then

$$\|\tilde{A}(t)\| \le \|A(t)\| \qquad (t \in J)$$

Remark 3. From the fact that the operators $\tilde{A}(t)$ and $\tilde{Q}_n$ commute with the projectors P_k $(k = 1, 2)$ it follows that the impulsive equation

$$\frac{dy}{dt} = R'R^{-1}y \qquad (t \bar{\in} \{t_n\}) \tag{13}$$

$$y(t_n^+) = R(t_n^+)R^{-1}(t_n)y(t_n) \tag{14}$$

can be split into a system of two equations, each for the respective subspace $X_k = P_kX$ $(k = 1, 2)$.

Corollary 1. Let the conditions of Theorem 2 hold, the operators being unitary, i.e., $Q_n^* = Q_n^{-1}$.

Then equation (1), (2) is kinematically similar to an equation without impulse effect, i.e., $\tilde{Q}_n = I$. If, moreover, the equality

$$A(t_n^+)Q_n = Q_n A(t_n) \qquad (t_n \in J)$$

is valid, then the operator-valued function $\tilde{A}(t)$ is continuous for $t \in J$.

Proof. From Theorem 2 it follows that the given equation is kinematically similar to equation (13), (14), where R is a positive square root of the operator R^2 defined by equality (9). Then

$$U^*(t_n^+)U(t_n^+) = U^*(t_n)Q_n^* Q_n U(t_n) = U^*(t_n)U(t_n)$$

Hence R^2, and therefore R as well, are continuous for $t \in J$, i.e., $\tilde{Q}_n = I$.

Let, moreover, $A(t_n^+)Q_n = Q_n A(t_n)$. Then

$$\begin{aligned}(U^*U)'|_{t=t_n^+} &= (U^*A^*U + U^*AU)|_{t=t_n^+} \\ &= U^*(t_n)Q_n^* A^*(t_n^+)Q_n U(t_n) + U^*(t_n)Q_n^* A(t_n^+)Q_n U(t_n) \\ &= U^*(t_n)A^*(t_n)Q_n^* Q_n U(t_n) + U^*(t_n)Q_n^* Q_n A(t_n)U(t_n) \\ &= (U^*A^*U + U^*AU)|_{t=t_n} = U^*U)'|_{t=t_n}\end{aligned}$$

The result obtained shows us that the function $(U^*U)'$ is continuous for $t \in J$, whence by formula (9) there follows the continuity of the function $[R^2(t)]'$. Formula (10) implies the continuity of the function R', and therefore of $\tilde{A} = R'R^{-1}$ as well.

Corollary 1 is proved. ■

Theorem 3. For equation (1), (2) let Conditions A and B hold and $J = [0, \infty)$.

Then the impulsive equation (1), (2) is kinematically similar to an impulsive operator whose operator and impulse operators are Hermitian.

Proof. The operator A can be represented in the form $A = A_1 + A_2$, where the operator $A_1 = \frac{1}{2}(A + A^*)$ is Hermitian and $A_2 = \frac{1}{2}(A - A^*)$ is skew-Hermitian, i.e., $A_2^* = -A_2$. The operator $Q_n^* Q_n$ is Hermitian and, moreover, for each element $z \in X$, $\|z\| = 1$, the estimate

$$(Q_n^* Q_n z, z) = \|Q_n z\|^2 \geq \frac{1}{\|Q_n^{-1}\|^2}, \qquad \|z\|^2 > 0$$

is valid, i.e., $Q_n^* Q_n$ is positive definite.

We represent the impulse operators Q_n in the form

$$Q_n = T_n (Q_n^* Q_n)^{1/2} \tag{15}$$

where the operators T_n are unitary.

Consider the impulsive equation

$$\frac{dy}{dt} = A_2(t)y \qquad (t \bar{\in} \{t_n\}) \tag{16}$$

$$y(t_n^+) = T_n y(t_n) \tag{17}$$

We shall show that the evolutionary operator $V(t)$ of this equation is unitary. For $t \neq t_n$ the following equalities are valid:

$$(V^*V)' = V^*A_2^*V + V^*A_2V = -V^*A_2V + V^*A_2V = 0$$

i.e., $V^*V = \text{const}$ on each interval $(t_n, t_{n+1}] \cap J$.

Moreover,

$$V^*(t_n^+)V(t_n^+) = V^*(t_n)T_n^*T_nV(t_n) = V^*(t_n)V(t_n)$$

i.e., $V^*V = \text{const}$ on the whole interval J. Since $V(0) = I$, then $V^*(t)V(t) \equiv V^*(0)V(0) = I$, i.e., the operator $V(t)$ $(t \geq 0)$ is unitary, hence it is bounded and has a bounded inverse one.

The operator-valued function $V^{-1}(t)$ transforms equation (1), (2) into an equation kinematically similar to it with operator $\tilde{A}$ and impulse operators $\tilde{Q}_n$ which we obtain according to formulas (5) and (6)

$$\tilde{A} = V^{-1}AV + (V^{-1})'V = V^*(AV - V') = V^*(AV - A_2V)$$
$$= \frac{1}{2}V^*(A + A^*)V$$

from which it immediately follows that the operator $\tilde{A}$ is Hermitian. For $\tilde{Q}_n$ we respectively obtain

$$\tilde{Q}_n = V^{-1}(t_n^+)Q_nV(t_n)$$

Since the operator-valued function $V(t)$ is a solution of the impulsive equation (16), (17), then $V(t_n^+) = T_nV(t_n)$, i.e.,

$$\tilde{Q}_n^* = V^*(t_n)Q_n^*T_nV(t_n) \tag{18}$$

From equality (15) and from the fact that the operator T_n is unitary there follows the validity of the equality

$$Q_n^*T_n = T_n^*Q_n \tag{19}$$

which implies $\tilde{Q}_n^* = \tilde{Q}_n$.

Theorem 3 is proved. ■

Corollary 2. If the impulse operators Q_n are unitary, then the impulsive equation (1), (2) is kinematically similar to an equation without impulses and with Hermitian operator.

Proof. Consider a kinematical similarity with transforming function $V^{-1}(t)$. Then from formulas (18) and (19) we obtain

$$\tilde{Q}_n^* = (V^{-1}(t_n)T_n^{-1}Q_n^{*-1}V^{*-1}(t_n))^{-1} = (V^*(t_n)T_n^*Q_nV(t_n))^{-1} = \tilde{Q}_n^{-1}$$

Corollary 3 is proved. ■

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